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Notes on ferrimagnetic resonance modes in a semi-infinite cylindrical boundary

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The magnetostatic solutions of ferrimagnetic resonance in a ferrite cylinder are briefly derived considering inhomogeneous field distribution inside the material. The resonance modes for $n = 0, 1, 2, 3, 4, 5$, for different values of Ω_H and Ω are calculated.

1. INTRODUCTION

Under applied non-uniform r f. fields, non-uniform distributions of magnetisation can be excited in the ferrite material. These distributions can be calculated, to a very good approximation by using the magnetostatic condition, $\nabla \times \boldsymbol{H} = 0$ and hence they are called magnetostatic modes. Ferrite material of different shapes are in use in technology and in various scientific experimentations It is, therefore, very useful to have a tabulation of the characteristic modes that are most likely to be encountered in the ferrite material of different boundary For example Fletcher & Bell (1959) has solved the magnetostatic problem of a sphere in a non-uniform r f. field and has given whereas possible, expressions for the resonant fields, potential functions, and r f magnetic moment. Walker (1957) has given the magnetostatic solutions of ferrimagnetic resonance for the case of spheroidal and elliptic boundaries. The assumptions made in solving the problem are : (i) The wavelength is small enough so that exchange effects are small, (ii) Non-linear effects are neglected i.e., r.f fields are small compared with saturation.

We have considered the case of a ferrite specimen of semi-infinite cylindrical boundary In a semi-infinite cylinder, however, it can be expected that the internal field will be markedly inhomogeneous, namely when the internal fields smaller than $4\pi \boldsymbol{M}$ are considered Consequently, it will be no more possible to describe the relation between the magnetization and the field by eq (4) that follows which is appropriate for a symmetrical body.

2. FORMULATION OF THE PROBLEM

Z-axis is taken along the axis of the cylinder and x, y, z form a right handed system. When the field distribution is homogeneous, the

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differential equation for magnetostatic potential inside the ferrite specimen is given by

$$(1+k)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\frac{\partial^2\psi}{\partial z^2}=0. \quad \ldots \quad (1)$$

The corresponding differential equation for the field outside is

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=0. \quad \ldots \quad (2)$$

This differential equation for ψ is valid when we consider the magnetic moment variation $m_z = 0$, which implies that H_z is constant along the z-axis. In order to introduce inhomogeneity we choose z-component of magnetisation M_z as consisting of a constant part M and a small perturbing part m_z which is a function of x and y i.e.

$$M_z = M+m_z(x,y) \quad \ldots \quad (3)$$

We have the gyromagnetic equation

$$\frac{1}{\gamma}\frac{\partial \boldsymbol{M}}{\partial t} = (\boldsymbol{M}\times\boldsymbol{H})$$

which is when written in terms of the components are

$$\begin{aligned} i\omega m_x &= \gamma[m_y H-(m_z+M)h_y], \\ i\omega m_y &= \gamma[(M+m_z)h_x - m_x H], \\ i\omega m_z &= \gamma[m_x h_y - h_x m_y], \end{aligned} \quad \ldots \quad (4)$$

where $h_z \simeq H$, from which we arrive approximately,

$$m_x = \frac{\gamma}{4\pi i\omega}\left[k\frac{\partial\psi}{\partial x}-i\nu\frac{\partial\psi}{\partial y}\right], \quad (4a)$$

$$m_y = \frac{\gamma}{4\pi i\omega}\left[i\nu\frac{\partial\psi}{\partial x}+k\frac{\partial\psi}{\partial y}\right], \quad \ldots \quad (4b)$$

$$m_z = \frac{\gamma}{4\pi i\omega}\left[\left\{k\frac{\partial\psi}{\partial x}-i\nu\frac{\partial\psi}{\partial y}\right\}\frac{\partial\psi}{\partial y}-\left\{i\nu\frac{\partial\psi}{\partial x}+k\frac{\partial\psi}{\partial y}\right\}\frac{\partial\psi}{\partial x}\right] \quad (4c)$$

where

$$k = \frac{\Omega_H}{(\Omega_H^2-\Omega^2)}, \quad \nu = \frac{\Omega}{(\Omega_H^2-\Omega^2)}, \quad \Omega_H = \frac{Hi}{4\pi M}, \quad \Omega = \frac{\omega}{4\pi\gamma M},$$

$$Hi = H_0 - \frac{4\pi M}{3}. \quad \ldots \quad (4d)$$

We have the final expressions for m_x, m_y, and m_z from eq (4) as,

$$m_x = \frac{\gamma}{i\omega}\left[\frac{1}{4\pi}\left\{i\nu\frac{\partial\psi}{\partial x}+k\frac{\partial\psi}{\partial y}\right\}\times H-\left\{M-\frac{\nu\gamma}{4\pi\omega}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\right\}\frac{\partial\psi}{\partial y}\right],$$

$$m_y = \frac{\gamma}{i\omega}\left[\left\{M-\frac{\nu\gamma}{4\pi\omega}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\right\}\frac{\partial\psi}{\partial x}-\frac{1}{4\pi}\left\{k\frac{\partial\psi}{\partial x}-i\nu\frac{\partial\psi}{\partial y}\right\}\right],$$

$$m_z = -\frac{\nu\gamma}{4\pi\omega}\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right] \qquad \dots \quad (5)$$

We have the field equations as div $\boldsymbol{H} = -4\pi$ div $\boldsymbol{M}$, when expressed in terms of ψ, becomes

$$\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right) = -4\pi\left[\frac{\partial}{\partial x}(m_x)+\frac{\partial}{\partial y}(m_y)+\frac{\partial}{\partial z}(m_z)\right] \qquad \dots \quad (6)$$

Substituting the values of m_x, m_y and m_z from eq. (5) we have the differential equation for ψ inside the material as

$$(1+k)\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]+\frac{\partial^2\psi}{\partial z^2}+\beta^2\frac{\partial}{\partial z}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0 \qquad \dots \quad (7)$$

where

$$k = \frac{\nu\gamma H}{\omega},\ \beta^2 = \frac{\nu\gamma}{\omega}. \qquad \dots \ (7a)$$

so that the eq. (1) is modified for the inhomogeneity as given by eq. (7) When eqs. (2) and (7) are transformed in cylindrical coordinates.

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2}=0 \text{ (for exterior region)} \qquad \dots \quad (8)$$

$$(1+k)\left[\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]+\frac{\partial^2\psi}{\partial z^2}+\beta^2\frac{\partial}{\partial z}\left[\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right]=0$$

$$\text{(for interior region)} \qquad \dots \quad (9)$$

The general solution for the eqs. (8) and (9) may be written in the forms

$$\left.\begin{aligned}\psi_{interior} &= \int_0^\infty A(\alpha')I_n(\alpha' r)e^{in\theta}\phi(z,\alpha')d\alpha',\\ \psi_{exterior} &= \int_0^\infty C(\alpha)K_n(\alpha r)e^{in\theta}\cos\alpha z\,d\alpha,\end{aligned}\right\} \qquad \dots \quad (10)$$

where

$$\left.\begin{aligned}\varphi(z,\alpha') &= \cos\alpha'\left\{1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right\}^{\frac{1}{2}}.Z.\\ \text{and}\qquad \alpha &= \alpha'\left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)^{\frac{1}{2}}.\end{aligned}\right\} \quad \ldots (11)$$

We have assumed that β is small such that

$$\exp.-\left(\frac{\beta^2 2\alpha'^2}{2(1+k)^2)}\right)\cos\ \alpha'z\times\left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)^{\frac{1}{2}}\simeq\cos\alpha'z\left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)^{\frac{1}{2}}$$

Obviously the solutions (10) satisfy the condition $\partial\psi/\partial z=0$ on $z=0$. There are two boundary condition corresponding to the (H) tangential and $(B)_{normal}$ conditions. Translated in terms of ψ, these become

$$\left.\begin{aligned}&\text{(i)}\qquad \psi_{interior}\Big|_{r=a}=\psi_{exterior}\Big|_{r=a}\\ &\text{(ii)}\qquad (1+k)\ \frac{\partial\psi_{int}}{\partial r}\Big|_{r=a}=\frac{\partial\psi_{ext}}{\partial r}\Big|_{r=a}\end{aligned}\right\} \quad \ldots (12)$$

From the first boundary condition, we have

$$\int_0^\infty C(\alpha)K_n(\alpha r)e^{in\theta}\cos(\alpha z)d\alpha=\int_0^\infty A(\alpha')I_n(\alpha' r)e^{in\theta}\cos\ \alpha'z\ \left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)^{\frac{1}{2}}d\alpha'$$

Taking cosine integral inversion,

$$C(\alpha)K_n(\alpha r)=\frac{2}{\pi}\int_0^\infty\cos\ \alpha z\,d\alpha\int_0^\infty A(\alpha')I_n(\alpha' r)\cos\left(\alpha' z\left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)^{\frac{1}{2}}\right)d\alpha'$$

$$=\int_0^\infty A\alpha(')I_n(\alpha' r)\delta\left(\alpha-\alpha'\left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)^{1/2}\right)d\alpha'$$

$$C(\alpha)K_n(\alpha a)=A(\alpha')I_n(\mathrm{p}'r) \quad \ldots (13)$$

where

$$\alpha=\alpha'\left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)^{1/2},$$

where δ is the Dirac delta function. From second boundary condition

$$\alpha[C(\alpha)]K'_n(\alpha a)=\int_0^\infty\alpha'(1+k)I'_n(\alpha' a)A(\alpha')\times\delta\left\{\alpha-\alpha'\left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)\right\}d\alpha' \quad \ldots (14)$$

From (13) and (14) we get as

$$\frac{C(\alpha)}{A(\alpha')} = \frac{I_n(\alpha' a)}{K_n(\alpha a)} = \frac{\alpha'}{\alpha}\left(\frac{(1+k)I'_n(\alpha' a)}{K'_n(\alpha a)}\right) \qquad \dots \quad (15)$$

From (15) we have the frequency equation as

$$(1+k)\alpha' I'_n(\alpha' a)K_n(\alpha a) - \alpha I_n(K'_n(\alpha a) = 0 \qquad \dots \quad (16)$$

The eq (17) can be written first in terms of α' from eq. (11) and then in terms of ν, γ, H and ω by putting the values of k and β^2 as,

$$\left(1+\frac{\nu\gamma H}{\omega}\right)\alpha' I'_n(\alpha' a)\times K_n\left\{\alpha'\left(1-\frac{\nu^2\gamma^2\alpha'^2}{4(\omega+\nu\gamma H)^2}\right)^{1/2}, a\right\}-$$

$$-\alpha'\left(1-\frac{\nu^2\gamma^2\alpha'^2}{4(\omega+\nu\gamma H)^2}\right)^{1/2}\times I_n(\alpha' a)K'_n\left\{\alpha'\left(1-\frac{\nu^2\gamma^2\alpha'^2}{4(\omega+\nu\gamma H)^2}\right)^{1/2} a\right\} = 0 \quad \dots \ (17)$$

When eq. (17) is written in terms of Ω and Ω_H and $\alpha' a$ in the form $\alpha'/M.aM$ and put $aM = 1$, we have

$$\left[1+\frac{(\Omega_H+1/3)}{(\Omega_H^2-\Omega^2)}\right]\times I'_n\left(\frac{\alpha'}{M}\right)\times K_n\left\{\frac{\alpha'}{M}\left(1-\frac{\alpha'^2}{64\pi^2M^2[(\Omega_H^2-\Omega^2)+(\Omega_H+1/3)]^2}\right)^{\frac{1}{2}}\right\}$$

$$-\left(1-\frac{\alpha'^2}{64\pi^2M^2[(\Omega_H^2-\Omega^2)+(\Omega_H+1/3)]^2}\right)^{\frac{1}{2}}\times$$

$$\times I_n\left(\frac{\alpha'}{M}\right)\times K'_n\left\{\frac{\alpha'}{M}\left(1-\frac{\alpha'^2}{64\pi^2M^2[(\Omega_H^2-\Omega^2)+(\Omega_H+1/3)]^2}\right)^{\frac{1}{2}}\right\} = 0 \qquad \dots \quad (18)$$

Since α corresponds to the wave number and α' is related to α by the relation $\alpha = \alpha'\left(1-\frac{\beta^4\alpha'^2}{4(1+k)^2}\right)^{\frac{1}{2}}$, we have in eq (17) the characteristic equation Expression (18), is the frequency equation for $aM = 1$, for different magnetostatic modes excited inside a ferrimagnetic semi-infinite cylinder when an external magnetic field is applied along the z-axis of the cylinder and the inhomogeneity of the internal field is considered.

We have numerically evaluated the values of α'/M for different values of Ω_H, Ω and for the different azimuthal modes $n = 0, 1, 2, \dots 5$ from the expression (22). For the numerical evaluation we have used the recurrence relations for $I_n(z)$, $K_n(z)$, $K'_n(z)$ and $I'_n(z)$ (c.f. Magnus & Oberhettinger 1954).

The expression (22) has given several number of roots of α'/M corresponding to different magnetostatic modes for every azimuthal modes. The curves α'/M

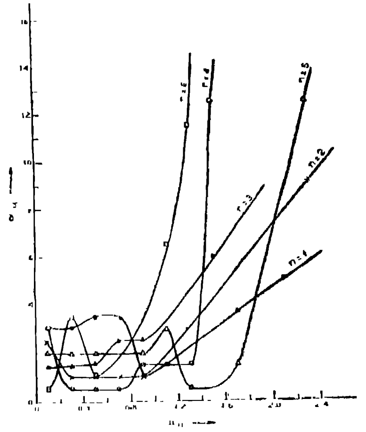

Fig 1 First modes corresponding to the first roots of the frequency equation at different azimuthal harmonics (n) i. e. $n=0, 1, 2, \ldots 6$ in a semi-infinite cylindrical ferrite.

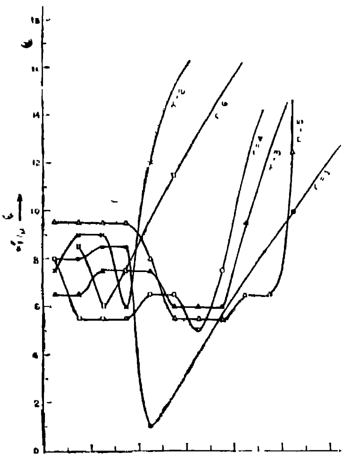

Fig 2. Third modes corresponding to the third roots of the frequency equation at different azimuthal harmonics i. e. for $n=0, 1, 2, \ldots, 6$ in a semi-infinite cylindrical ferrite material.

against Ω_H are plotted for different modes which represent the dispersion curves for the medium.

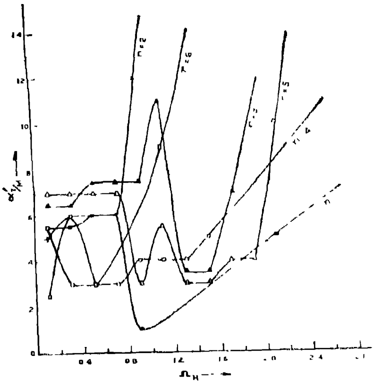

Fig. 3 Second modes corresponding to the second roots of the frequency equation at different azimuthal harmonics (n) i e. for $n=0$, to 6 in a semi-infinite cylindrical ferrite material.

The different numerical values of α'/M for a few values of Ω_H and Ω are given in Table 1

Table 1

n	Ω_H	Ω	α'/M					
	0.1	9.0	3 08466,	5 50315,	8.05976,	10.50099,		
	0.3	10.0	0 50099.	3 07138,	5.50083,	8 00235,	10.50214,	
	0 5	11.0	1 00215,	3.50235.	6,00212,	8,51523,	11.00112,	
0	0.7	12.0	1.00102,	3 49782,	6.00518,	8 57621.	11.00582,	
	0.9	13.0	1.00521,					
	2.1	19 0	5.06806,	7.53214,	10.00321,	12 58921,		
	2.3	20.0	2.50021.	5 00321,	7 58234,	10 00762,	12.51002,	
	0.1	9 0	2.51009,	5.00211.	7 50021,	10.00325,		
	0.3	10.0	1.00319,	3 50068,	6.00109,	8.50092,	11.00925,	
	0.5	11.0	1.00229,	3.52124,	6.00892,	8 59921.	11.01087,	
1	0.7	12.0	1.00208,	3.58726,	6.00987,			
	0.9	13.0	1.00209,	12.00576,				
	1.1	14.0	4.50073,	7.00524,	9 50021,	12 00562,		
	2 3	20.0	9.10005,	11.50006,				
	1 0	9.0	1.49875,	4.00871,	6 52381,	9.00287,	11.52538,	11 52538
	0 3	10.0	1.49008,	4.00125,	6 50001,	9.00001,	11.50042,	
	0.5	11.0	1 50000;	5.00021,	7.50008,	10.00042,	12.50062,	
	0 7	12.0	2.50000,	5.00871.	7.50003,	10.00002,	12.50001,	
	0 9	13.0	2.50012,	5.00028,	7.50001,	10.00021,		
2	1.1	14.0	1 00081,	3.50008,	6.00042,	8.50002,	11.00089,	
	1.3	15.0	1.00092,	2.50087,	6.00081,	8.50008,	11.00072.	
	1.5	16.0	1.00072,	3.50009,	6.00008,			
	1.7	17.0	4.50008,	7.00032,	9.51002,	12.01231,		
	1.9	18.0	2.00072,	4.50087,	7.00032,	9 52321,	12.00042,	
	2.1	19.0	2.00005,					

Table 1 (Contd)

n	Ω_H	Ω	α'/M					
	0 1	9 0	3.00087,	5.50032,	8.00072,	10.50002,		
	0 3	10.0	0.50032,	3.00021,	5.50210,	8.00071,	10.50008,	
	0 5	11 0	0 50001.	3.00002.	5 50021,	8.00007,	10.50002,	
3	0 7	12 0	0 50008,	3 00008,	5 50003,	8.00001.	9 00042,	11.500071
	0.9	13.0	1.50042,	4 00072,	6.50032,	9 00028.	11.50007,	
	1 1	14 0	1 50004,	4 00002.	6 52001,	9 00007,	11.50001,	
	1 3	15.0	1 50006,	4 00008,	5 00042,	7 50032,	10 00021,	12 507021
	1 5	16.0	2.50021.	5 00072,	7 50028,	10.00032,	12.50008,	
	0 1	9 0	1 50032,	7 00021,	9.50004,	12.00032,		
	0 3	10 0	4 50002,	7.00004.	9 50001,	12.00042,		
	0 5	11.0	4.50029.	7.00218.	9.50004.	12 00287,		
	0 7	12 0	1.50001,	7 00032,	9.52007.	12 00087.		
	0 9	13 0	2 00872.	3.00721.				
4	1 0	14 0	3 00081,	5 55231,	8 00762,	10.50042,		
	1 3	15.0	0 50021.	3 00021,	5 50021,	8 00072.	10.50092.	
	1.5	16 0	0 50087,	3.00072.	5 50072.	8 00092,	11 50021,	
	1 7	17.0	1 50081,	4.00032,	6 50082,	9 00723.	11 50081,	
	1.9	18.0	1.50072,	4 00071,	6.50004,			
	2 1	19.0	7.50087,	10 000821,	12.50063,			
	2 3	20 0	2 50007,	5 00042,	7 50021.	10.00042.	12 50003,	
	0 1	9 0	0.50021,	2 50072,				
	0 3	10 0	3.50072,	6.00081,	8 50004,	11 00004,		
5	0 5	11 0	1.00072,	3.50008,	6 00032,	8.50004.	11.00032,	
	1.1	14 0	6 50002,	9 00042,	11 50004,			
	1 3	15 0	1 50004,	4 00001,	6 50003,	9 00082,	11 50004,	

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